INTRO TO GROUP THEORY - FEB. 29, 2012
PROBLEM SET 4 - GT4. NORMAL SUBGROUPS AND QUOTIENT GROUPS

1. In S_3, let $H_1 = \{e, (12)\}, H_2 = \{e, (13)\}$, and $H_3 = \{(e, (123), (132)\}$. Verify that $HH = H$ in each case, and compute H_1H_2, H_2H_1, H_1H_3, and H_3H_1.

2. (a) Suppose H is a subgroup of G and $N \triangleleft G$. Show that HN is a subgroup of G.

(b) Suppose $N \subset H$. Show that H/N is a subgroup of G/N. If H' is a subgroup of G/N, find a subgroup H such that $H' = H/N$.

(c) Suppose $H \triangleleft G$. Show that $H/N \triangleleft G/N$. Find a non-normal subgroup H and normal subgroup N in D_8 such that $H/N \triangleleft D_8/N$.

3. (a) Find an example of $K \triangleleft H$, $H \triangleleft G$, but K is not normal in G.

(b) Show that the intersection of normal subgroups is a normal subgroup.

4. Verify that $Z(A_4) = \{e\}$ and that $Z(D_8) = \{e, (13)(24)\}$.

5. (a) Let G be finite group with subgroup H. Explain why it is sufficient to check the normal condition for H using a set of generators for G.

(b) Noting that $S_3 = \langle (12), (23) \rangle$, find all normal subgroups of S_3.

(c) Noting that $A_4 = \langle (123), (12)(34) \rangle$, find all normal subgroups of A_4.

(d) Noting that $D_8 = \langle (1234), (14)(23) \rangle$, find all normal subgroups of D_8.

(e) Noting that $S_4 = \langle (12), (23), (34) \rangle$, show that A_4 is normal in S_4. Explain what each coset measures.

6. (a) Suppose $N \triangleleft G$. If x has finite order in G, show that $|xN|$ (as a group element in G/N) divides $|x|$.

(b) Let $H = \{e, (12)(34), (13)(24), (14)(23)\}$. Find the orders of each element in A_4/H and $D_8/Z(D_8)$.

7. If G is a cyclic group, show that all quotient groups of G are cyclic. Find all quotient groups for $\mathbb{Z}/n$.

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8. (Linear algebra) Recall that $O(2)$ is the subgroup of $GL(2,\mathbb{R})$ consisting of orthogonal 2×2 matrices A: $A^{-1} = A^T$. Also $SO(2)$ is the subgroup of $O(2)$ consisting of orthogonal matrices A with $det(A) = 1$.

(a) Is $O(2)$ normal in $GL(2,\mathbb{R})$? Is $O(2)$ abelian?

(b) Show that $SO(2) \triangleleft O(2)$. What is the order of $O(2)/SO(2)$? Find a representative for each coset.

(c) Each element of $SO(2)$ can be written in the form $\begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix}$. Use this to show that $SO(2)$ is abelian. If A is in $O(2)$ but not $SO(2)$, give a similar representation. Describe both cases geometrically.

(d) Redo Problem Set 3, Problem 7(b) with $GL(2,\mathbb{R})$ and $O(2)$.

9. (Linear algebra) Let $G = GL(2,\mathbb{R})$, and let $H = SL(2,\mathbb{R})$ be the subgroup of real invertible 2×2 matrices A with $det(A) = 1$.

(a) Show that $H \triangleleft G$. Is $SO(2) \triangleleft SL(2,\mathbb{R})$?

(b) Show that $Trace(gAg^{-1}) = Trace(A)$. Find other matrix quantities preserved by conjugation by elements in G.

10. Let T be the torsion subset of G; that is, T is the subset of all elements of finite order in G.

(a) If G is abelian, show that $T \triangleleft G$.

(b) If G is non-abelian, show that T is preserved under conjugation by elements of G. Is T a subgroup?

(c) Find T for $\mathbb{R}/\mathbb{Z}$. Describe $(\mathbb{R}/\mathbb{Z}) - T$ and $(\mathbb{R}/\mathbb{Z})/T$.

(d) Find T for the multiplicative groups $\mathbb{R}^*$ and $\mathbb{C}^*$ (nonzero real and complex numbers).