INTRO TO GROUP THEORY - APR. 25, 2012
SOLUTION SET 12
GT19/20. CAUCHY'S THEOREM/ SYLOW THEORY

1. (a) A_4 and S_4 have no elements of order 6, S_4 has no element of order 12.

(b) A_4 has no subgroup of order 6 (check conjugacy classes), A_5 has no subgroup of order 20 or 30.

2. All Sylow p-subgroups are conjugate, and conjugation yields an isomorphism between subgroups. Likewise, conjugation carries normalizers to normalizers isomorphically.

If $N(H_p)$ contains more than one Sylow p-subgroup for G, then p divides $[G : N(H_p)]$, which implies that H_p is not of maximal order. If g normalizes $N(H_p)$ then g normalizes H_p, the unique subgroup of its order in $N(H_p)$. Thus g is in H_p.

3. Two conjugacy classes: the classes must be the identity class $\{e\}$ and a class of elements of prime order p. Thus G has order p^k and $|Z(G)| \geq p$. Thus $p = 2$ and $k = 1$. So $G \cong \mathbb{Z}/2$.

Three conjugacy classes: if G is a p-group, then $|Z(G)| \geq p$. Thus $G \cong \mathbb{Z}/3$. Otherwise there are exactly two classes with element orders given by primes p and q with $p < q$. The Sylow q-subgroup H is normal (since unique) and a union of conjugacy classes of G. Thus $q^k = 1 + p^j$, which implies $p = 2$ and q is odd. Let $Z(H)$ be the center of the Sylow q-subgroup; $Z(H)$ is characteristic in H. By Cauchy's Theorem, there exists an element x of order 2, and a subgroup generated by x and $Z(H)$. Since orbits by a subgroup of order 2 have one or two elements, $q = 3$ and $|G| = S_3$.

4. (a) S_4: 24 elements, four Sylow 3-subgroups, and three Sylow 2-subgroups. A typical Sylow 3-subgroup is $\{e, (123), (132)\}$ with normalizer S_3. A Sylow 2-subgroup is given by D_8 and is its own normalizer. To count the Sylow 2-subgroups, count the elements of order 4; D_8 has exactly 2 elements of order 4.

S_5: 120 elements, 6 Sylow 5-subgroups, 10 Sylow 3-subgroups, and 15 Sylow 2-subgroups. A typical Sylow 5-subgroups is $\{e, (12345), (13524), (14253), (15432)\}$, which has normalizer $\langle (12345), (2354) \rangle$ with order 20. The Sylow 2-subgroups are also isomorphic to D_8 and self-normalize, and the normalizer of each Sylow 3-subgroup is isomorphic to $S_3 \times \mathbb{Z}/2$.

A_6 : 360 elements, 36 Sylow 5-subgroups, 10 Sylow 3-subgroups, and 45 Sylow 2-subgroups. The Sylow 3-subgroups are of the form $\mathbb{Z}/3 \times \mathbb{Z}/3$, with normalizers generated

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by elements of $S_3 \times S_3$ in A_6 and an element of order four that switches triplets; for example, $N(H_3) = \langle (123), (456), (12)(45), (1425)(36) \rangle$ (order 36). Otherwise the same as S_5. Note that the D_8 subgroups consist of elements of $D_8 \times \mathbb{Z}/2$ in A_6 and normalize themselves. Each is the centralizer of some $(ab)(cd)$. To count the Sylow 3-subgroups, there are 40 three-cycles and 40 elements of the form $(abc)(def)$; the Sylow 3-subgroups intersect trivially.

S_6: 720 elements, 36 Sylow 5-subgroups, 10 Sylow 3-subgroups, and 45 Sylow 2-subgroups. Same as A_6, save that the Sylow 2-subgroups are isomorphic to $D_8 \times \mathbb{Z}/2$. The normalizer of a Sylow 3-subgroup is isomorphic to $\langle S_3 \times S_3, (ad)(be)(cf) \rangle$ and the normalizer of a Sylow 2-subgroup is itself.

(b) First there exist 4 Sylow 3-subgroups. The normalizer of a Sylow 3-subgroup H has $|G|/|N(H)| = 6$ elements, and its normalizer is itself. If this normalizer has an element of order 6, then there are $2 \times 4 = 8$ elements of order 6. Thus the Sylow 2-subgroup is normal since it has eight elements. With our assumptions, the normalizer of a Sylow 3-subgroup is isomorphic to S_3.

Now G acts by conjugation on both the Sylow 3-subgroups and their normalizers. Considering the action on the normalizers, there is a homomorphism of G into S_4. If g acts trivially, then g normalizes each S_3. Thus g is in each $N(H_3)$ and must have order 1, 2, or 3. The intersection of any two of these normalizers has at most two elements. If the intersection of all normalizers is trivial, then the kernel is trivial and $G \cong S_4$. Otherwise the kernel is a normal subgroup with two elements and G has an element of order 6 by considering the semidirect product of any H_3 with this kernel.

(c) $A_4 \times \mathbb{Z}/2$ has 8 elements of order 6 of the form $((123), 1)$. See also $SL(2, \mathbb{Z}/3)$ below.

(d) As noted above, if G has 4 Sylow 3-subgroups and an element of order 6, then there are 8 element of order 3, eight elements of order 6, and the Sylow 2-subgroup is normal. By (b), G is not isomorphic to S_4.

We show that a unique Sylow 3-subgroup H implies an element of order 6. In this case, H is normal. Since $Aut(\mathbb{Z}/3) \cong \mathbb{Z}/2$, the Sylow 2-subgroup contains a four element group that centralizes H. Thus an element x of order 2 centralizes H, and the subgroup generated by x and H is isomorphic to $\mathbb{Z}/6$.

5. Order 12: if no normal Sylow p-subgroup, then there are 4 Sylow 3-subgroups and 3 Sylow 2-subgroups. Thus there are 8 elements of order 3 and a unique Sylow 2-subgroup, a contradiction.

Order 30: if none, then there are 6 Sylow 5-subgroups and 10 Sylow 3-subgroups. These yield 24 elements of order 5 and 20 elements of order 10, a contradiction.

Order 40, 45: The Sylow 5-subgroup is unique, so normal.

Order 42: The Sylow 7-subgroup is unique, so normal.

6. (a) D_{30} : the unique Sylow 3- and 5-subgroups are subgroups of the rotation group R_{15}. There are 15 Sylow 2-subgroups, each generated by a reflection.

D_{60} : again the unique Sylow 3- and 5-subgroups are subgroups of the rotation group R_{30}. There are 15 Sylow 2-subgroups, each generated by the central element r^{15} and a reflection. These are isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$.

(b) For each prime $p \neq 2$ that divides n, the Sylow p-subgroup is unique and contained in R_n. The Sylow 2-subgroups are isomorphic to a dihedral group if they have more than 4 elements.

If n is odd, there exist n Sylow-2 subgroups of order 2, each generated by a reflection.

If $n = 2^k m, k > 1, m$ odd, then there are m Sylow 2-subgroups of order 2^{k+1}. In fact, these are all conjugates of the subgroup $H = \langle c, r^m \rangle$, which is isomorphic to $D_{2^{k+1}}$. These conjugates are of the form $\langle cr^{2j}, r^m \rangle$.

7. (a) The Sylow 7-subgroup is unique and normal. There are 7 Sylow 3-subgroups, each equal to its normalizer.

(b) Suppose $p < q$. Then the Sylow q-subgroup is unique and normal. If G is abelian, then the Sylow p-subgroup is also unique and normal. Otherwise $q = pk + 1$ and there are k Sylow p-subgroups, each equal to its own normalizer.

8. (a) Check that G does not divide $[G : H]!$ for some Sylow p-subgroup H.

(b) Since $p^k < q$, the Sylow q-subgroup is unique and thus normal.

9. (a) This is a tedious check, using either Sylow Theory, including counting elements, or the Corollary to Cayley's Theorem. See Problems 4 and 7.

(b) Using the techniques in (a), we are left with orders 72, 112, 120, and 144. Burnside's Theorem resolves 72, 112, and 144. Order 120 remains.

10. (a) The order of G is $(q-1)q(q+1)$. Any three consecutive integers contain an even integer and a multiple of three. So Cauchy's Theorem says there exists element of order 2, 3, and q; we also give an element of order 4:

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The normalizer of the Sylow q-subgroup consists of upper triangular matrices. In G, these have $q(q-1)$ elements, with x and x^{-1} on diagonals. Thus there are $(q+1)$ Sylow q-subgroups.

(b) Since all determinants equal 1, the characteristic polynomials for each conjugacy class are of the form $p_A(x) = x^2 + kx + 1$ with k in $\mathbb{Z}/q$. We can find representatives for each class using companion matrices or Jordan form. This can be used to count elements or perform deeper analysis.

$q = 3$: G has 24 elements. By (a), there are 4 Sylow 3-subgroups and their normalizers are cyclic of order 6. The Sylow 2-subgroup is unique by counting. Since this group has elements of order 4 and 6, it is isomorphic to neither S_4 or $A_4 \times \mathbb{Z}/2$. It has 4 Sylow 3-subgroups and the unique Sylow 2-subgroup is isomorphic to Q, the quaternion group:

$$\pm I, \qquad \pm\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \qquad \pm\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \qquad \pm\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

with orders 1(2), 4, 4, and 4. An element of order 6 is given by $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$.

$q = 5$: G has 120 elements. There are 6 Sylow 5-subgroups (for 24 elements of order 5) and their normalizers are cyclic of order 10 (24 elements of order 10). The normalizer of a Sylow 3-subgroup has 12 elements, so there are 10 Sylow 3-subgroups (20 elements); this also gives 20 elements of order 6 using $-I$.

$q = 7$: G has 336 elements. There are 8 Sylow 7-subgroups (for 48 elements). Checking solutions to $x^3 = 1$ using $p_A(x)$ shows that there is a single conjugacy class of elements of order 3. Computing the centralizer (tedious) shows 6 elements, so 56 elements of order 3. Thus we have 28 Sylow 3-subgroups. Further (tedious) checking of conjugacy classes shows 21 Sylow 2-subgroups with 16 elements.

11. (a) Since there are no normal subgroups, there are 8 Sylow 7-subgroups (possibilities are divisors of 24). Call one H_7. Then the normalizer $N(H_7)$ has 21 elements. Since any two Sylow 7-subgroups intersect trivially, there are $8 \times 6 = 48$ elements of order 7. If we show that $N(H_7)$ is non-abelian, then there are no elements of order 21. There are no elements of order 14; otherwise $N(H_7)$ would have an element of order 2.

There are 7 or 28 Sylow 3-subgroups (possibilities are divisors of 56). If $N(H_7)$ were abelian then 7 would divide $N(H_3)$, the normalizer of a Sylow 3-subgroup, and 7 could not be a divisor of the number of Sylow 3-subgroups. So $N(H_7)$ is non-abelian and contains 7 Sylow 3-subgroups. This also shows that there are two conjugacy classes of order 7 elements, each with 24 elements.

(b) Consider the intersection of two Sylow-7 normalizers. This subgroup is either trivial or order 3, so there are more than 7 Sylow 3-subgroups. Thus there are 28 Sylow 3-subgroups.

Now there are $28 \times 2 = 56$ elements of order 3, and the normalizer of a Sylow 3-subgroup has $168/28 = 6$ elements. If $N(H_3)$ were cyclic of order 6, there would be $2 \times 28 = 56$ elements of order 6, and $48 + 56 + 56 = 160$ elements with orders 7, 3, or 6. Then the Sylow 2-subgroup would be unique and normal, a contradiction. So $N(H_3) \cong S_3$, and there is a single conjugacy class of 56 elements with order 3. No element of order 6 rules out elements of order 12 and 24.

(c) We show there are 21 Sylow 2-subgroups isomorphic to D_8. There remain 63 elements of order 2, 4, or 8. If there were 3 or 7 Sylow 2-subgroups, the counts would fall short. So there are 21 Sylow 2-subgroups, and the normalizer of a Sylow 2-subgroup H_2 is itself. Counting shows that at least two such H_2 intersect non-trivially.

Every group of order 8 has a central element x of order 2. Since there are no elements of order 6 or 14, its centralizer has order 8, and its conjugacy class has 21 elements.

If H_2 were abelian, the centralizer of any non-trivial element in an intersection of H_2's would have more than eight elements, a contradiction. So H_2 is non-abelian (Q or D_8) and must have elements of order 4. Thus there is a conjugacy class consisting of 42 elements with order 4 and no elements of order 8.

All that remains is to identify the isomorphism class of $H_2 = N(H_2)$. As noted, two normalizers cannot intersect in their central elements. If H_2 is isomorphic to the quaternionic group Q, a non-trivial intersection means they intersect in at least the unique element of order 2, which is central. If all intersections are trivial, then there are 6×21 elements of order 4, a contradiction. So $H_2 \cong D_8$.

(d) Consider the subgroup of matrices of the form

$$\begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{pmatrix}.$$

The *abcd* block is isomorphic to S_3, so there are $6 \times 2 \times 2 = 24$ elements. There are no elements of order 6, so $G \cong S_4$.