INTRO TO GROUP THEORY - MAY 2, 2012
SOLUTION SET 13
GT21/22. INTERNAL PRODUCTS/ FINITE ABELIAN GROUPS

1. Order 12: See Problem 2.

Order 15: The Sylow subgroups are unique, so H_3 and H_5 intersect trivially, generate G, and are both normal. Thus $G \cong \mathbb{Z}/3 \times \mathbb{Z}/5 \cong \mathbb{Z}/15$.

Order 20: The Sylow 5-subgroup is unique and normal. If the Sylow 2-subgroup H_2 is also unique, then G is abelian since all groups of order 4 are abelian and G is a direct product of its Sylow subgroups. In this case, G is isomorphic to $\mathbb{Z}/20$ or $\mathbb{Z}/2 \times \mathbb{Z}/10$. Otherwise there are five Sylow 2-subgroups H_2. Since any H_2 and H_5 intersect trivially and generate the group, we have a semidirect product and it is enough to consider non-trivial homomorphisms from $\mathbb{Z}/4$ or $\mathbb{Z}/2 \times \mathbb{Z}/2$ into $Aut(\mathbb{Z}/5) \cong \mathbb{Z}/4$.

If we use the two automorphisms on $\mathbb{Z}/4$ (corresponding to squaring or cubing in $Aut(\mathbb{Z}/5)$), we get the semidirect product $y^4 = e, x^5 = e, yxy^{-1} = x^2$ (rep. x^3) This group is realized as the subgroup $\langle (12345), (2354) \rangle$ in S_5 (resp. $\langle (12345), (2453) \rangle$). See also the affine group $Aff(\mathbb{Z}/5)$. If we let $z = y^3$, then G also satisfies the relations $z^4 = e, x^5 = e, zxz^{-1} = x^3$. Thus we obtain only one isomorphism class in this manner. The order two automorphism yields the semidirect product

$$y^4 = e,\ x^5 = e,\ yxy^{-1} = x^4.$$

Otherwise any nontrivial homomorphism from $H_2 = \mathbb{Z}/2 \times \mathbb{Z}/2$ into $\mathbb{Z}/4$ will yield $\mathbb{Z}/2 \times D_{10} \cong D_{20}$. The kernel of this homomorphism has two elements, say $\{e, x\}$. To finish, let c be another generator of H_2 and r any generator of H_5. For the isomorphism to D_{20}, use c and rx as generators.

Order 30: If we show that there exists a subgroup of order 15, then it is unique and normal. Since there exists an element of order 2, we need to consider elements of order 2 in $Aut(\mathbb{Z}/15) \cong (\mathbb{Z}/15)^* \cong \mathbb{Z}/2 \times \mathbb{Z}/4$. There are four of these, and each lead to one of $\mathbb{Z}/30$, D_{30}, $S_3 \times \mathbb{Z}/5$, and $\mathbb{Z}/3 \times D_{10}$. The non-abelian types can be distinguished by counting elements of order 2.

Now there are either 1 or 6 Sylow 5-subgroups, but the Sylow 3-subgroup is unique and normal. Thus there exists a subgroup of order 15.

Order 63: The Sylow 7-subgroup is unique and normal. If the Sylow 3-subgroup is also unique, then G is isomorphic to either $\mathbb{Z}/3 \times \mathbb{Z}/21$ or $\mathbb{Z}/63$. Otherwise we consider non-trivial homomorphisms from $\mathbb{Z}/9$ or $\mathbb{Z}/3 \times \mathbb{Z}/3$ into $Aut(\mathbb{Z}/7) \cong \mathbb{Z}/6$. Now $\mathbb{Z}/6$ contains

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a unique $\mathbb{Z}/3$ subgroup, so we are essentially looking at the non-trivial automorphism of $\mathbb{Z}/3$. In the second case, we obtain $\mathbb{Z}/3 \times G_{21}$, where G_{21} is the non-abelian group of order 21 (unique up to isomorphism).

The first case is new; we have the relations $y^9 = e, x^7 = e, yxy^{-1} = x^2$. The elements of order 3 in $Aut(\mathbb{Z}/7)$ correspond to squaring and fourth powers. There are 7 Sylow 3-subgroups H_3, which give $7 \times \phi(9) = 42$ elements of order 9, and the normalizer of a Sylow 3-subgroup is itself. Since any group of order 9 is abelian, the normalizer of a three element subgroup is either H_3 or G itself. If H_3, then each three element subgroup in a given H_3 is distinct and there are 14 elements of order 3. By the relations, there exist elements of order 21, and counting shows that the three element subgroup is unique and normal. So G has e, six elements of order 7, 42 elements of order 9, two elements of order 3, and $\phi(21) = 12$ elements of order 21. The last count follows since G contains a unique cyclic subgroup of order 21. Again we have only one isomorphism class since G also satisfies the relations $z^9 = e, x^7 = e, zxz^{-1} = x^4$ when $z = y^2$.

2. We have seen five isomorphism classes so far: $\mathbb{Z}/12$, $\mathbb{Z}/2 \times \mathbb{Z}/6$, A_4, D_{12}, and the semidirect product G_{12} described by the relations $y^4 = e, x^3 = e, yxy^{-1} = x^{-1}$. These can be distinguished by the abelian property and counting elements of order 4.

There are either 1 or 4 Sylow 3-subgroups. If four, there are 8 elements of order 3 and the Sylow 2-subgroup is unique and normal. Since any Sylow 3-subgroup and the Sylow 2-subgroup generate G, G is a semidirect product of $\mathbb{Z}/3$ and $\mathbb{Z}/2 \times \mathbb{Z}/2$; since $Aut(\mathbb{Z}/4) \cong \mathbb{Z}/2$, the Sylow 2-subgroup is not $\mathbb{Z}/4$. Note that conjugation permutes the four Sylow 3-subgroups; if we check the Class Equation, the center of G is trivial, so $G \cong A_4$, the unique subgroup of order 12 in S_4.

If G contains only one Sylow 3-subgroup, it is normal. Thus G is a semidirect product using either $\mathbb{Z}/4$ or $\mathbb{Z}/2 \times \mathbb{Z}/2$. If the Sylow 2-subgroup H_2 is also unique, then G is isomorphic to either $\mathbb{Z}/12$ or $\mathbb{Z}/2 \times \mathbb{Z}/6$. If not, we define the multiplication using non-trivial homomorphisms from H_2 into $Aut(H_3)$. When $H_2 \cong \mathbb{Z}/4$, we obtain G_{12}, and, when $H_2 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, we obtain $\mathbb{Z}/2 \times S_3 \cong D_{12}$.

3. If G is non-abelian and $q = pk + 1$, then there exists a unique Sylow q-subgroup H_q and k Sylow p-subgroups H_p. Since H_q and H_p generate G and intersect non-trivially, G is a semidirect product of $\mathbb{Z}/p$ and $\mathbb{Z}/q$. We have seen that this group is unique up to isomorphism, and is the unique subgroup of order pq in $Aff(\mathbb{Z}/q)$. In this case, $Z(G) = \{e\}$. If there were a non-trivial central element of order p or q, we could represent G as a direct product of cyclic groups to find G abelian.

Otherwise G is abelian as a direct product of abelian groups and isomorphic to $\mathbb{Z}/pq$. Thus $Z(G) = G$.

4. Order 18: Sylow Theory shows that G has a subgroup H_3 of order 9, which must be normal by the Index Two Theorem. If $H_3 \cong \mathbb{Z}/9$, the G is isomorphic to $\mathbb{Z}/18$ or D_{18}

(note that $Aut(\mathbb{Z}/9) \cong \mathbb{Z}/6$). If $H_3 \cong \mathbb{Z}/3 \times \mathbb{Z}/3$, then G is isomorphic to either $\mathbb{Z}/3 \times \mathbb{Z}/6$, $\mathbb{Z}/3 \times S_3$, or a dihedral-like semidirect product

$$G_{18} = \langle y, H_3 \rangle \text{ with } y^2 = e,\ yxy^{-1} = x^{-1} \quad \text{for } x \in H_3.$$

To distinguish the non-abelian types, count elements of order 6 or 9. For examples, note the subgroup of S_6 given by $\langle (123), (456), (14)(25)(36) \rangle \cong \mathbb{Z}/3 \times S_3$ with 6 elements of order 6. The subgroup of A_6 given by $\langle (123), (456), (12)(45) \rangle$ is isomorphic to G_{18} with no elements of order 6 or 9.

Order 24: There are 15 isomorphism classes, so a bit tedious. First suppose G has 4 Sylow 3-subgroups H_3. Conjugation of these subgroups gives a non-trivial homomorphism from G into S_4. We show this kernel has at most 2 elements. If trivial then $G \cong S_4$; if non-trivial, then there exists a normal subgroup with two elements and there exist elements of order 6. We have seen that a group of order 24 with no element of order 6 is isomorphic to S_4.

Suppose x is in the normalizer of each H_3. Since each $N(H_3)$ has 6 elements, the intersection of any two has at most 2 elements, and the assertion follows.

Suppose we have a normal subgroup H of order 2. H is contained in $Z(G)$ by considering conjugates. In turn, any central element would normalize the above subgroups, so $H = Z(G)$. We count 8 elements of order 3 and 8 elements of order 6, so the Sylow 2-subgroup H_2 is unique and normal. Since G is not a direct product of H_3 and H_2, $Aut(H_2)$ must contain an element of order 3. The groups of order eight are of isomorphic to one of $\mathbb{Z}/8,\ \mathbb{Z}/2 \times \mathbb{Z}/4, \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2, D_8$ or Q, the quaternion group. The only possibilities are $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ and Q; the automorphism groups are isomorphic to $SL(3, \mathbb{Z}/2)$ (order 168) and S_4 (order 24). In the first case, we obtain $\mathbb{Z}/2 \times A_4$, and, in the second, $SL(2, \mathbb{Z}/3)$. (3 classes)

Suppose G has a unique Sylow 3-subgroup H_3. Suppose H_2 is cyclic of order 8. If H_2 is unique, then G is isomorphic to $\mathbb{Z}/24$. Otherwise G_{24} is the semidirect product defined by the relations $y^8 = e, x^3 = e, yxy^{-1} = x^2$. G_{24} has center $\langle y^2 \rangle$. Since there are 3 Sylow 2-subgroups, there are $3 \times \phi(8) = 12$ elements of order 8 of the form $\{y^{odd}x^i\}$. There are also 2 elements of order 4 of the form $\{y^2, y^6\}$, 4 elements of order 12 of the form $\{y^2x^i, y^6x^i\}$, and 2 elements of order 6 of the form $\{y^4x^i\}$. y^4 is the unique element of order 2. (2 classes)

Suppose H_2 is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/4$. If H_2 is unique, then $G \cong \mathbb{Z}/2 \times \mathbb{Z}/12$. Otherwise there are two non-trivial homomorphisms into $Aut(\mathbb{Z}/3) \cong \mathbb{Z}/2$. Thus G is isomorphic to either $\mathbb{Z}/4 \times S_3$ or $\mathbb{Z}/2 \times G_{12}$. To distinguish, the first type has elements of order 12 while the second does not. (3 classes)

Suppose H_2 is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$. Then G is isomorphic to either $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/6$ or $\mathbb{Z}/2 \times \mathbb{Z}/2 \times S_3 \cong \mathbb{Z}/2 \times D_{12}$. (2 classes)

Suppose H_2 is isomorphic to D_8. If H_2 is unique, then $G \cong \mathbb{Z}/3 \times D_8$. Otherwise there are three homomorphisms from D_8 to $Aut(\mathbb{Z}/3)$ with kernels R_4, $\{e, (13), (24), (13)(24)\}$, or $\{e, (12)(34), (14)(23), (13)(24)\}$. Since an automorphism interchanges the last two subgroups, we obtain only three classes in total, including D_{24}. (3 classes)

Suppose H_2 is isomorphic to Q. If unique, then $G \cong \mathbb{Z}/3 \times Q$. Otherwise each non-trivial homomorphism from H_2 to $Aut(\mathbb{Z}/3)$ has a cyclic subgroup of order 4 in its kernel, say $\langle i \rangle$, and we have a semidirect product with relations $x^3 = e, ix = xi, jxj^{-1} = x^{-1}$. (2 classes)

Order 28: The Sylow 7-subgroup is unique and normal, and there are 1 or 7 Sylow 2-subgroups. In the first case, G is abelian and isomorphic to either $\mathbb{Z}/28$ or $\mathbb{Z}/2 \times \mathbb{Z}/14$. Otherwise G is a semidirect product, and we consider homomorphisms from $\mathbb{Z}/4$ or $\mathbb{Z}/2 \times \mathbb{Z}/2$ into $Aut(\mathbb{Z}/7) \cong \mathbb{Z}/6$. The latter case gives $\mathbb{Z}/2 \times D_{14} \cong D_{28}$.

In the former case, we have the semidirect product G_{28} with relations

$$y^4 = e, x^7 = e, yxy^{-1} = x^6.$$

The unique element of order 2 in $Aut(\mathbb{Z}/7)$ is multiplication by $6 = -1$. Again we have a cyclic subgroup $\langle x, y^2 \rangle$ of order 14. Since G has 7 Sylow 2-subgroups isomorphic to $\mathbb{Z}/4$, there are $7 \times 2 = 14$ elements of order 4 of the form $\{y^{odd}x^i\}$. Note that

$$(y^{odd}x^i)^4 = y^{4\ odd} = e.$$

5. First we count $48 = 3 \times 16$ elements. Since there exists an element of order 12 and none of larger order, G is isomorphic to either $\mathbb{Z}/4 \times \mathbb{Z}/12$ or $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/12$. Since there are three elements of order 2, FTFAG says there are at most 2 even factors. So $G \cong \mathbb{Z}/4 \times \mathbb{Z}/12$.

6. Order 16: $\mathbb{Z}/16,\ \mathbb{Z}/8 \times \mathbb{Z}/2, \mathbb{Z}/4 \times \mathbb{Z}/4, \mathbb{Z}/4 \times \mathbb{Z}/2 \times \mathbb{Z}/2, \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$.

Order $200 = 8 \times 25$: $\mathbb{Z}/200, \mathbb{Z}/2 \times \mathbb{Z}/100, \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/50, \mathbb{Z}/2 \times \mathbb{Z}/10 \times \mathbb{Z}/10,$ $\mathbb{Z}/5 \times \mathbb{Z}/40, \mathbb{Z}/10 \times \mathbb{Z}/20$.

To check, there are 6 total classes: 3 possible Sylow 2-subgroup and 2 possible Sylow 5-subgroups.

Order $360 = 5 \times 8 \times 9$: $\mathbb{Z}/360, \mathbb{Z}/2 \times \mathbb{Z}/180, \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/90, \mathbb{Z}/2 \times \mathbb{Z}/6 \times \mathbb{Z}/30,$ $\mathbb{Z}/3 \times \mathbb{Z}/120, \mathbb{Z}/6 \times \mathbb{Z}/60$.

7. $Aut(\mathbb{Z}/48) \cong (\mathbb{Z}/48)^* \cong (\mathbb{Z}/3 \times \mathbb{Z}/16)^* \cong (\mathbb{Z}/3)^* \times (\mathbb{Z}/16)^* \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/4$.

$Aut(\mathbb{Z}/72) \cong (\mathbb{Z}/72)^* \cong (\mathbb{Z}/8 \times \mathbb{Z}/9)^* \cong (\mathbb{Z}/8)^* \times (\mathbb{Z}/9)^* \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/6$.

$Aut(\mathbb{Z}/100) \cong (\mathbb{Z}/100)^* \cong (\mathbb{Z}/4 \times \mathbb{Z}/25)^* \cong (\mathbb{Z}/4)^* \times (\mathbb{Z}/25)^* \cong \mathbb{Z}/2 \times \mathbb{Z}/20$.

8. We've seen before that no n has $\phi(n) = 3$. So $n = 7, 9, 14,$ or 18. For the product of two $\mathbb{Z}/6$ factors, we can multiply these if relatively prime. So $n = 63, 126$. No n yields three factors.

9. (a) $\mathbb{Z}/6 : 0 + 1 + 2 + 3 + 4 + 5 = 15 = 3,$

$\mathbb{Z}/7: \ 15 + 6 = 21 = 0$,
$\mathbb{Z}/2 \times \mathbb{Z}/2 : (0,0) + (1,0) + (0,1) + (1,1) = (2,2) = (0,0)$,

$$\begin{aligned}\mathbb{Z}/2 \times \mathbb{Z}/4 : (0,0) &+ (0,1) + (0,2) + (0,3) + (1,0) + (1,1) + (1,2) + (1,3) \\ &= (4,12) = (0,0).\end{aligned}$$

(b) If G has odd order, then there are no elements of order 2 by Lagrange's Theorem. Thus every non-identity element occurs in an inverse pair, and these pairs cancel in the sum. So the sum is the identity.

(c) If G has even order, part (b) still applies, but the sum now equals the sum of the elements of order 2. With the identity, these elements form a subgroup isomorphic to a product of $\mathbb{Z}/2$ factors. If only one factor occurs, then the sum is the unique element of order 2 in G; see $\mathbb{Z}/6$. Otherwise the number of factors equals the number of even factors using FTFAG. Inductively we see the sum over these is zero; if S_{n-1} is the sum with $n-1$ factors, then the new sum has two parts: $(0, S_{n-1})$ and $(2^{n-1}, S_{n-1})$. See $\mathbb{Z}/2 \times \mathbb{Z}/2$ and $\mathbb{Z}/4 \times \mathbb{Z}/2$.

10. We can represent $H = \mathbb{Z}/2 \times \mathbb{Z}/4$ in terms of generators and relations as

$$y^4 = e, x^2 = e, xy = yx.$$

There are four elements of order 4 (y, y^3, xy, xy^3), three elements of order 2 (x, y^2, xy^2), and the identity. An automorphism is determined on x and y; we may send y to any element of order 4, but this choice eliminates the image of y^2 from the possibilities for x. So there are at most eight automorphisms, and it straightforward to check that each works. If one enumerates these, there are 5 automorphisms of order 2, so the isomorphism class is D_8.

11. (a) Note that many ideas from linear algebra apply here if we are careful with the zero divisor 2. By definition, any automorphism π preserves the additive property:

$$\pi(g + h) = \pi(g) + \pi(h).$$

If we apply this repeatedly with $g = h$, we obtain the scalar property:

$$\pi(kg) = k\pi(g), \qquad k \in \mathbb{Z}/4.$$

So every automorphism corresponds to a $\mathbb{Z}/4$-linear map, and, using the basis $\{(0,1),(1,0)\}$, we associate these maps with 2×2 matrices with entries in $\mathbb{Z}/4$.

Since the maps are invertible, the associated matrices are invertible. The usual equation for classical adjoint applies here: $A\, cl(A) = det(A)I$. That is, A will have an inverse if and only if $det(A) \neq 0, 2$. One can also check this directly using the formula for 2×2 matrices.

(b) First we consider the elements of H. There are 12 twelve elements of order 4, three elements of order 2, and the identity; the elements of order 4 have at least one coordinate equal to 1 or 3. The element of order 12 separate into three sets: each element of order 2 has exactly four square roots. For instance, $(2,0)$ is a square of $(1,0),(1,2),(3,0)$ and $(3,2)$. Elements of G are determined on $(1,0)$ and $(0,1)$, both which have order 4. If we assign $(1,0)$ to the order four element (a,b) then $(2,0)$ maps to $(2a,2b)$. Then $(0,1)$ cannot be assigned to an element with square $(2a,2b)$, and there are at most $8\times 12=96$ automorphisms of H.

On the other hand, consider the usual group action of G on H as matrix-vector multiplication. We consider the orbit of $e_1=(1,0)$. We can carry e_1 to any other element v of order 4 by an element T of G; set $T(e_1)=v$ and $T(e_2)=e_1$ or e_2, whichever makes $det(T)=1,3$. Thus the orbit of e_1 has 12 elements. By observation, the stabilizer of e_1 in G is the set of upper triangular matrices of the form $\begin{pmatrix}1 & a\\ 0 & b\end{pmatrix}$, which has 8 elements. Thus $|G|=8\times 12=96$.

(c) In H, the characteristic polynomials are of the form $p_A(x)=x^2+kx+1$ where $k=-tr(A)=3tr(A)$. To count, we find centralizers of elements by solving $AX=XA$.

First consider $p_A(x)=x^2+1$. There is a single conjugacy class for this polynomial with 12 elements of order 4:

$$\pm\begin{pmatrix}0&1\\3&0\end{pmatrix},\pm\begin{pmatrix}1&3\\2&3\end{pmatrix},\pm\begin{pmatrix}1&2\\1&3\end{pmatrix},\pm\begin{pmatrix}2&3\\1&2\end{pmatrix},\pm\begin{pmatrix}3&2\\1&1\end{pmatrix},\pm\begin{pmatrix}1&1\\2&3\end{pmatrix}.$$

Next we consider $p_A(x)=x^2+3x+1$. Again we have a single class with 8 elements of order 6:

$$\begin{pmatrix}0&1\\3&1\end{pmatrix},\begin{pmatrix}1&3\\1&0\end{pmatrix},\begin{pmatrix}3&1\\1&2\end{pmatrix},\begin{pmatrix}2&3\\3&3\end{pmatrix},\begin{pmatrix}2&1\\1&3\end{pmatrix},\begin{pmatrix}3&3\\3&2\end{pmatrix},\begin{pmatrix}1&1\\3&0\end{pmatrix},\begin{pmatrix}0&3\\1&1\end{pmatrix}.$$

To obtain the conjugacy class for $p_A(x)=x^2+x+1$, we multiply the previous class by $3I$ to obtain 8 elements of order 3:

$$\begin{pmatrix}0&3\\1&3\end{pmatrix},\begin{pmatrix}3&1\\3&0\end{pmatrix},\begin{pmatrix}1&3\\3&2\end{pmatrix},\begin{pmatrix}2&1\\1&1\end{pmatrix},\begin{pmatrix}2&3\\3&1\end{pmatrix},\begin{pmatrix}1&1\\1&2\end{pmatrix},\begin{pmatrix}3&3\\1&0\end{pmatrix},\begin{pmatrix}0&1\\3&3\end{pmatrix}.$$

For $p_A(x)=x^2+2x+1$, we obtain five conjugacy classes: the two central classes $\{I\},\{3I\}$, and three classes with six elements of orders 2, 4, and 4. First the order 2 class:

$$\begin{pmatrix}1&2\\0&1\end{pmatrix},\begin{pmatrix}3&2\\0&3\end{pmatrix},\begin{pmatrix}3&2\\2&3\end{pmatrix},\begin{pmatrix}1&0\\2&1\end{pmatrix},\begin{pmatrix}3&0\\2&3\end{pmatrix},\begin{pmatrix}1&2\\2&1\end{pmatrix}.$$

Then the two order 4 classes differ by multiplication by $3I$:

$$\begin{pmatrix}1&1\\0&1\end{pmatrix},\begin{pmatrix}1&3\\0&1\end{pmatrix},\begin{pmatrix}2&3\\1&0\end{pmatrix},\begin{pmatrix}1&0\\1&1\end{pmatrix},\begin{pmatrix}1&0\\3&1\end{pmatrix},\begin{pmatrix}0&1\\3&2\end{pmatrix},$$

and
$$\begin{pmatrix}3&3\\0&3\end{pmatrix},\begin{pmatrix}3&1\\0&3\end{pmatrix},\begin{pmatrix}2&1\\3&0\end{pmatrix},\begin{pmatrix}3&0\\3&3\end{pmatrix},\begin{pmatrix}3&0\\1&3\end{pmatrix},\begin{pmatrix}0&3\\1&2\end{pmatrix}.$$
Summing we have $12+8+8+1+1+6+6+6=48$.

(d) Now consider characteristic polynomials of the form $p_A(x)=x^2+kx+3$ with $k=-Tr(A)=3Tr(A)$.

Both x^2+x+3 and x^2+3x+3 contribute a single conjugacy class each with eight elements of order 6, interchanged by multiplication by $3I$:
$$\begin{pmatrix}0&1\\1&3\end{pmatrix},\begin{pmatrix}3&1\\1&0\end{pmatrix},\begin{pmatrix}1&3\\1&2\end{pmatrix},\begin{pmatrix}2&3\\1&1\end{pmatrix},\begin{pmatrix}0&3\\3&3\end{pmatrix},\begin{pmatrix}3&3\\3&0\end{pmatrix},\begin{pmatrix}1&1\\3&2\end{pmatrix},\begin{pmatrix}2&1\\3&1\end{pmatrix};$$
$$\begin{pmatrix}1&1\\1&0\end{pmatrix},\begin{pmatrix}0&1\\1&1\end{pmatrix},\begin{pmatrix}2&3\\1&3\end{pmatrix},\begin{pmatrix}3&3\\1&2\end{pmatrix},\begin{pmatrix}1&3\\3&0\end{pmatrix},\begin{pmatrix}0&3\\3&1\end{pmatrix},\begin{pmatrix}2&1\\3&3\end{pmatrix},\begin{pmatrix}3&1\\3&2\end{pmatrix}.$$
Likewise $p_A(x)=x^2+2x+3$ contributes a single class of twelve elements of order 4:
$$\pm\begin{pmatrix}0&1\\1&2\end{pmatrix},\pm\begin{pmatrix}1&2\\3&1\end{pmatrix},\pm\begin{pmatrix}1&2\\1&1\end{pmatrix},\pm\begin{pmatrix}2&1\\1&0\end{pmatrix},\pm\begin{pmatrix}1&3\\2&1\end{pmatrix},\pm\begin{pmatrix}1&1\\2&1\end{pmatrix}.$$

Finally $p_A(x)=x^2+3$ yields three classes of 2, 6, and 12 elements of order 2:
$$\pm\begin{pmatrix}3&2\\2&1\end{pmatrix};\qquad\pm\begin{pmatrix}3&0\\0&1\end{pmatrix},\pm\begin{pmatrix}3&2\\1&0\end{pmatrix},\pm\begin{pmatrix}3&0\\2&1\end{pmatrix};$$
$$\pm\begin{pmatrix}0&1\\1&0\end{pmatrix},\pm\begin{pmatrix}1&3\\0&3\end{pmatrix},\pm\begin{pmatrix}3&3\\0&1\end{pmatrix},\pm\begin{pmatrix}1&0\\3&3\end{pmatrix},\pm\begin{pmatrix}1&0\\1&3\end{pmatrix},\pm\begin{pmatrix}2&3\\3&2\end{pmatrix}.$$

(e) The order of H is $48=3\times16$. Counting elements of order 3 gives 4 Sylow 3-subgroups $H_3\cong\mathbb{Z}/3$. Counting elements of order 1, 2, and 4 gives $12+6+6+6+1+1=32$ elements, so there must be 3 Sylow 2-subgroups of order 16. In this case, $H_2=N(H_2)$ is isomorphic to $\mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/4$. For instance, consider $H_2=\langle\begin{pmatrix}a&b\\0&a\end{pmatrix},\begin{pmatrix}1&2\\2&1\end{pmatrix}\rangle$. $N(H_3)$ has 12 elements, including elements of order 4 and 6. Since there are no elements of order 12, $N(H_3)\cong G_{12}$. In fact, the centralizer of H_3 in H is cyclic with 6 elements and the remaining elements have order 4.

Now $96=3\times32$. Since G has the same number of element of order 3 as H, there are 4 Sylow 3-subgroups. There are 64 elements of order 1, 2, and 4, so again we have 3 Sylow 2-subgroups of order 32. Again $H_2=N(H_2)$ is isomorphic to $\mathbb{Z}/2\times\mathbb{Z}/2\times D_8$, represented by $\langle\begin{pmatrix}a&b\\0&c\end{pmatrix},\begin{pmatrix}1&2\\2&1\end{pmatrix}\rangle$.

$N(H_3)$ has 24 elements and its Sylow 3-subgroup is unique. A direct check shows that the centralizer of H_3 is isomorphic to $\mathbb{Z}/2\times\mathbb{Z}/6$. Since $\begin{pmatrix}2&1\\3&2\end{pmatrix}$ normalizes $\langle\begin{pmatrix}0&1\\3&1\end{pmatrix}\rangle$ but is not in its centralizer, $N(H_3)$ is the semidirect product with relations

$$y^4 = e, x^6 = e, yxy^{-1} = x^{-1}.$$

Its Sylow 2-subgroup is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/4$, and, from Problem 4, $G \cong \mathbb{Z}/2 \times G_{12}$.

(f) $Inn(H) \cong H/Z(H)$ has 24 elements. We check for elements of order 6. Cubing any element from the class of order 6 elements gives $3I$, so there are no elements of order 6 in the quotient, and $Inn(H) \cong S_4$.

12. No candidates arise checking isomorphism classes up to order 15. We consider groups of order 16.

Consider $G_1 = \mathbb{Z}/4 \times \mathbb{Z}/4$ and $G_2 = \mathbb{Z}/2 \times Q$. In both cases, there is an identity, three elements of order 2, and twelve elements of order 4. These groups are not isomorphic since G_1 is abelian and G_2 is not.