INTRO TO GROUP THEORY - MAR. 7, 2012
PROBLEM SET 5 - GT5/6/7. INDEX 2 THEOREM, ETC.

1. Consider the four diagonals $\{A, B, C, D\}$ through antipodal vertices. We obtain the 6 four cycles by rotating about squares, which also give the 3 products of disjoint two cycles. We obtain the 8 three cycles by rotating about vertices. We obtain the 6 two cycles by rotating halfway in a plane that bisects the cube diagonally though four vertices. Note that we can inscribe a pair of tetrahedra inside the cube by using diagonals on opposite squares, rotated by half. The two and four cycles interchange the tetrahedra.

A_4 consists of the identity, three cycles, and products of disjoint two cycles. These rigid motions preserve the tetrahedra.

2. Let G be the dihedral group D_{2p} with $2p$ elements. Let $H = \{e, c\}$. Since $rcr^{-1} = cr^{-2}$, H is not normal and $[G : H] = p$.

3. (a) D_{10}: Rotations: $e, (12345), (13524), (14253), (15432)$;

Reflections, each fixes a single vertex: $(25)(34), (13)(45), (15)(24), (12)(35), (14)(23)$.

D_{16}: Rotations: $e, (12345678), (1357)(2468), (14725836),$
$(15)(26)(37)(48), (16385274), (1753)(2864), (18765432)$;

Reflections, fixing opposite vertices: $(28)(37)(46), (13)(48)(57), (15)(24)(68), (15)(24)(68)$;

Reflections, fixing opposite edges: $(12)(38)(47)(56), (23)(14)(58)(67),$
$(34)(25)(16)(78), (45)(36)(27)(18)$.

(b) $D_{2n} = \langle c, cr\rangle$. No. In D_{16}, $\langle cr^2, cr^4\rangle = \langle c, r^2\rangle$.

(c) Any reflection can be written in the form $R = \begin{pmatrix} cos(\theta) & sin(\theta) \\ sin(\theta) & -cos(\theta)\end{pmatrix}$, which has eigenvalues ± 1. Let v be a nonzero eigenvector with eigenvalue -1, and let v' be a nonzero eigenvector with eigenvalue 1. Since R is an orthogonal transformation, $\{v, v'\}$ is an orthogonal basis for $\mathbb{R}^2$. Thus it is enough to check the formula on the basis:

$$s_v(v) = v - 2\frac{\langle v, v\rangle}{\langle v, v\rangle} v = -v, \qquad s_v(v') = v' - 2\frac{\langle v', v\rangle}{\langle v, v\rangle} v = v'.$$

Geometrically this fixes the line perpendicular to v and switches the direction of the line along v through the origin. For R, the axis of reflection is along $(cos(\theta/2), sin(\theta/2))$; to see

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this, we note that R switches e_1 and Re_1 so the half angle is fixed. This should be verified using trig identities.

(d) Denote the rotation counter-clockwise by θ as $r(\theta)$. Note that $cr(\theta)$ is a reflection for all θ; that is, $cr(\theta)$ is orthogonal, not a rotation, and

$$cr(\theta)^2 = cr(\theta)cr(\theta) = c^2 r(\theta - \theta) = e.$$

Now $ccr(\theta) = r(\theta)$, so we obtain any rotation as a product of two reflections. Conversely, the product of any two reflections is a rotation:

$$(cr(\theta))(cr(\theta')) = c^2 r(-\theta) r(\theta') = r(\theta' - \theta).$$

4. (a) We verify the even case: In cycle notation,
   (1) $r = (123\dots n)$,
   (2) $c = (1n)(2\ n-1)(3\ n-2)\dots(n/2\ n/2+1)$, and
   (3) $r^{n/2} = (1\ n/2+1)(2\ n/2+2)\dots(n/2\ n)$.

Since c and r generate D_{2n}, it is enough to check

$$r[(1\ n/2+1)(2\ n/2+2)\dots(n/2\ n)]r^{-1} = (1\ n/2+1)(2\ n/2+2)\dots(n/2\ n)$$

and

$$c[(1\ n/2+1)(2\ n/2+2)\dots(n/2\ n)]c = (1\ n/2+1)(2\ n/2+2)\dots(n/2\ n).$$

If we verify that $crc = r^{-1}$ in cycle notation, the old argument holds. This is straightforward.

(b) Consider D_{2n} as matrices acting on $\mathbb{R}^2$. If x is in $Z(D_{2n})$ then x commutes with every element in the matrix span of D_{2n}. Now the span contains I and $diag(1,-1)$. It also contains $r - r^{-1}$, which is skew-symmetric, and $cr - cr^{-1}$, which is symmetric. Thus the span of D_{2n} is all of $M_2(\mathbb{R})$, and the central elements must be multiples of the identity, confirming the odd and even cases.

5. (a) Straightforward.

(b) Since A_4 has no subgroups of order 6 and $H \subseteq Z(H)$ in this case, $|Z(H)| = 3, 12$. Since $(12)(34)$ is not in $Z(H)$, $Z(H) = H$. A similar argument shows $Z(H) = H$ in S_4. On the other hand, $N(H)$ is S_3 since (12) is in $N(H)$ and $N(H) \neq A_4$.

In S_5, we augment $Z(H)$ and $N(H)$ with the element (45). Now $Z(H)$ is cyclic of order 6, generated by $(123)(45)$, and $N(H)$ is $S_3 \times \mathbb{Z}/2$, generated by (12) and $(123)(45)$. In turn, this group satisfies the relations for D_{12}.

(c) $Z(x) \subseteq N(H)$ is immediate. If h is in $N(H)$, then $heh^{-1} = e$ and $hxh^{-1} = x$. If not, $hxh^{-1} = e$, but then $x = e$. Thus $xh = hx$ and h is in $Z(x)$.

6. $\mathbb{Z}/p, \mathbb{Z}/2p$, and $\mathbb{Z}/pq$, respectively.

If G is abelian of order pq, then Lagrange's Theorem restricts the orders of subgroups and elements. Since there are nonidentity elements, there must be an element of order p, q, or

pq. Suppose $|x| = p$, and $H = \langle x \rangle$. Then G/H has q elements, and we choose y in G such that yH generates G/H. Now y has order pq or q since y^q is in H and $gcd(p, q) = 1$. If pq, $G = \langle y \rangle$, and G is cyclic. Otherwise consider the subgroup $K = \langle x, y \rangle$. Since pq divides $|K|$, we have $K = G$. This means $G = \langle xy \rangle$ and G is cyclic of order pq.

7. Order 8: $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$, $\mathbb{Z}/2 \times \mathbb{Z}/4$, $\mathbb{Z}/8$.
Order 12: $\mathbb{Z}/2 \times \mathbb{Z}/6$, $\mathbb{Z}/12$.

8. D_{10}: the commutator subgroup consists of all rotations; in the abelianization, the cosets correspond to rotation and reflection sets of elements.

D_{16}: the commutator subgroup consists of all rotations by multiples of $\frac{\pi}{2}$; in the abelianization, the cosets correspond to the set of rotations $r(\frac{k\pi}{2})$, to the set of rotations $r(\frac{(2k+1)\pi}{4})$, to the set of reflections with a fixed vertex pair, and to the set of reflections with a fixed edge pair.

9. A_4: for $\sigma\omega\sigma^{-1}$, the Conjugation Rule says to relabel using σ. Thus

$$(123)(124)(132)(142) = (234)(142) = (12)(34)$$

and

$$(12)(34)(123)(12)(34)(132) = (214)(132) = (13)(24).$$

This means $H = [A_4, A_4] = \{e, (12)(34), (13)(24), (14)(23)\}$, and the abelianization has three elements, so $\mathbb{Z}/3$.

S_4: H is contained in $[S_4, S_4]$. Now

$$(12)(123)(12)(132) = (213)(132) = (123),$$

so A_4 is contained in $[S_4, S_4]$. But $[S_4 : A_4] = 2$ and S_4/A_4 is abelian with two elements. Thus $A_4 = [S_4, S_4]$, and the abelianization is $\mathbb{Z}/2$. Note that this shows the quotient group S_4/H is S_3, since $\mathbb{Z}/2$ is the largest abelian quotient group of S_4.

S_5 : For now, brute force. We can augment the S_4 result to obtain A_5, the subgroup consisting of (even) permutations with structure $e, (abc), (ab)(cd), (abcde)$. A_5 has 60 elements, so $[S_5 : A_5] = 2$. Later we will see that A_5 has no proper normal subgroups; since the commutator subgroup is normal, it must be all of A_5.

10. (a) Since $SO(2)$ is abelian, $[SO(2), SO(2)] = \{e\}$. In any commutator for $O(2)$, the c terms occur in pairs and cancel if present. Consider

$$[c, cr(\theta)] = c(cr(\theta))c(r(-\theta)c) = r(2\theta).$$

Thus $[O(2), O(2)] = SO(2)$, and the abelianization is

$$O(2)/SO(2) = \{SO(2), cSO(2)\} \cong \mathbb{Z}/2.$$

(b) The computations will be identical, save for position. We show the upper triangular case. Then

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \begin{pmatrix} 1/a & -b/ac \\ 0 & 1/c \end{pmatrix} \begin{pmatrix} 1/x & -y/xz \\ 0 & 1/z \end{pmatrix} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix},$$

where d can be any real number. In this case, $G/[G,G]$ is the group $\mathbb{R}^* \times \mathbb{R}^*$; cosets are represented by the diagonal matrices.

(c) Since $det(ghg^{-1}h^{-1}) = 1$, both commutator subgroups are contained in $SL(2,\mathbb{R})$. If we show $[SL(2,\mathbb{R}), SL(2,\mathbb{R})] = SL(2,\mathbb{R})$, the result also follows for $GL(2,\mathbb{R})$.

Since every A in $GL(2,\mathbb{R})$ factors as RU where R is in $O(2)$ and U is upper-triangular, one rechecks the argument to show that every A in $SL(2,\mathbb{R})$ factors with R in $SO(2)$ and U upper-triangular with positive diagonal entries x and x^{-1}. By (a) and (b), it is enough to show that every diagonal matrix $diag(x, x^{-1})$ with $x > 0$ is in the commutator. For $a > 0$, we have

$$\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1/a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} a^2 & 0 \\ 0 & 1/a^2 \end{pmatrix}.$$

Thus the abelianization of $SL(2,\mathbb{R})$ is trivial, and the abelianization of $GL(2,\mathbb{R})$ is $\mathbb{R}^*$; cosets are represented by matrices $diag(det(A), 1)$.